Physics G8069
homework #5
due Mon. Oct. 16

1. **Symmetry breaking in finite volume?**

   Consider the quantum mechanics of a particle moving in a double well potential, described by the Lagrangian

   $$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{1}{4}m\lambda x^4 \,.$$

   We're taking the parameter ω^2 to be negative.

   (i) Expand the action to quadratic order about the two minima of the potential, and write down approximate (harmonic oscillator) ground state wavefunctions

   $$\begin{aligned} \Psi_+(x) &= \langle x|+\rangle \\ \Psi_-(x) &= \langle x|-\rangle \end{aligned}$$

   describing states $|+\rangle$ and $|-\rangle$ localized in the right and left wells, respectively. How do your wavefunctions behave as $m \to \infty$?

   (ii) Use the WKB approximation to estimate the tunneling amplitude $\langle -|+\rangle$. You can make approximations which are valid for large m (equivalently small λ).

   Now consider a real scalar field with Lagrangian ($\mu^2 < 0$)

   $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4 \,.$$

   The $\phi \to -\phi$ symmetry is supposed to be spontaneously broken, with two degenerate ground states $|+\rangle$ and $|-\rangle$. But can't the field tunnel from one minimum to the other? To see what's happening consider the same theory but in a finite spatial volume. For simplicity let's work in a spatial box of volume V with periodic boundary conditions, so that we can expand the field in spatial Fourier modes

   $$\phi(t,\mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^* \,.$$

(iii) Expand the field theory action to quadratic order about the classical vacua.

(iv) Use the actions worked out in part (iii) to write down approximate ground state wavefunctions

$$\begin{array}{ll} \Psi_+(\phi_{\mathbf{k}}) & \text{describing } |+\rangle \\ \Psi_-(\phi_{\mathbf{k}}) & \text{describing } |-\rangle \end{array}$$

How do your wavefunctions behave in the limit $V \to \infty$?

(v) If you neglect the coupling between different Fourier modes – something which should be valid at small λ – then the action for the constant mode ϕ_0 should look familiar. Use your quantum mechanics results to estimate the tunnelling amplitude $\langle -|+\rangle$ between the two (unit normalized) ground states. How does your result behave as $V \to \infty$?

Moral of the story: spontaneous symmetry breaking is a phenomenon associated with the thermodynamic ($V \to \infty$) limit. For a nice discussion of this see Weinberg QFT vol. II sect. 19.1.