Supersymmetry in 3+1 Physics G8070
Spring 2007

References: All this material can be found in Wess & Bagger. I'll just quote the main results. I use Wess & Bagger conventions but denote Lorentz vector indices μ, ν rather than m, n.

1.1 Superspace and superfields

Superspace is the collection of points $z^M = \left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right)$ where

x^μ	real bosonic coordinates ($\mu =$ Lorentz vector index)
θ_α	complex Grassmann coordinates ($\alpha =$ left-handed spinor index)
$\bar{\theta}_{\dot{\alpha}}$	complex conjugates of θ_α ($\dot{\alpha} =$ right-handed spinor index)

Supersymmetry is a translation

$$\begin{aligned}
&x^\mu \to x^\mu - i\bar{\xi}\bar{\sigma}^\mu\theta - i\xi\sigma^\mu\bar{\theta} \\
&\theta_\alpha \to \theta_\alpha + \xi_\alpha \\
&\bar{\theta}_{\dot{\alpha}} \to \bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}
\end{aligned}$$

where $\xi_\alpha, \bar{\xi}_{\dot{\alpha}}$ are Grassmann spinor parameters. This translation is generated by differential operators $Q_{\partial\alpha}$, $\bar{Q}_{\partial\dot{\alpha}}$.

$$\begin{aligned}
&\delta z^M = \left(\xi^\alpha Q_{\partial\alpha} + \bar{\xi}_{\dot{\alpha}}\bar{Q}_\partial^{\dot{\alpha}}\right) z^M \\
&Q_{\partial\alpha} = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \\
&\bar{Q}_\partial^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_\alpha\frac{\partial}{\partial x^\mu}
\end{aligned}$$

We can define some supercovariant derivatives D_α, $\bar{D}_{\dot{\alpha}}$ which anticommute with all the Q_∂'s, and therefore map superfields to superfields.

$$\begin{aligned}
&D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \\
&\bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_\alpha\frac{\partial}{\partial x^\mu}
\end{aligned}$$

A general (or unconstrained) superfield is an arbitrary complex function on superspace $F(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. Under a supersymmetry transformation we have

$$\delta F = \left(\xi Q_\partial + \bar{\xi}\bar{Q}_\partial\right) F\,.$$

An unconstrained superfield provides a reducible representation of supersymmetry. To find irreducible representations we need to impose some constraints on F that are preserved under supersymmetry.

1.2 Chiral superfields

A chiral superfield Φ is a superfield that satisfies $\bar{D}_{\dot{\alpha}}\Phi = 0$. It has the component expansion

$$\Phi = \phi(x) + \sqrt{2}\theta^\alpha\psi_\alpha(x) + \theta\theta F(x) + \cdots$$

where ϕ is a complex scalar field, ψ is a left-handed spinor field, F is a complex scalar auxiliary field, and $\cdots$ denotes extra terms built from derivatives of these fields. The supersymmetry transformations of the component fields are

$$\begin{aligned}
\delta\phi &= \sqrt{2}\xi^\alpha\psi_\alpha \\
\delta\psi_\alpha &= i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu\phi + \sqrt{2}\xi_\alpha F \\
\delta F &= i\sqrt{2}\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha
\end{aligned}$$

The complex conjugate of a chiral superfield is an antichiral superfield $\bar{\phi}$ satisfying $D_\alpha\bar{\Phi} = 0$.

You can write an invariant action by integrating an arbitrary superfield $\int d^4x d^4\theta$, or by integrating a chiral superfield $\int d^4x d^2\theta$. The standard kinetic term for a chiral superfield is

$$\begin{aligned}
S_{\text{kin}} &= \int d^4x d^4\theta\, \bar{\Phi}\Phi \\
&= \int d^4x\, -\partial_\mu\phi^*\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F
\end{aligned}$$

You can add a superpotential

$$\begin{aligned} S_{pot} &= \int d^4 d^2\theta\, W(\Phi) + \text{c.c.} \\ &= \int d^4x\, W'(\phi)F - \frac{1}{2}W''(\phi)\psi\psi + \text{c.c.} \end{aligned}$$

The superpotential has to be a holomorphic function of Φ, i.e. it can only depend on Φ not on Φ^*. Eliminating the auxiliary field we have the potential for the scalar field

$$\mathcal{V}(\phi,\phi^*) = F^*F = \left|\frac{\partial W}{\partial \phi}\right|^2 .$$

1.3 Abelian vector superfields

An abelian vector superfield V is a superfield that is constrained to be real, $V^* = V$. We're going to impose gauge invariance under transformations

$$V \to V + \Lambda + \bar{\Lambda}$$

where Λ is a chiral superfield. In "Wess-Zumino" gauge a vector superfield has the component expansion

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)$$

where A_μ is a real vector field, λ_α is a spinor field, and D is a real scalar auxiliary field. In WZ gauge you're just left with conventional gauge transformations of A_μ, namely

$$A_\mu \to A_\mu + \partial_\mu\omega(x)$$

with λ_α, D invariant. One can define a gauge-invariant "field strength" for V, namely

$$W_\alpha = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}D_\alpha V .$$

One can show that W_α is a chiral superfield. It lets us write a gauge-invariant and supersymmetric action for V.

$$\begin{aligned} S &= \int d^4x d^2\theta\, \frac{1}{4}W^\alpha W_\alpha + \text{c.c.} \\ &= \int d^4x\, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2 . \end{aligned}$$

1.4 Super Yang-Mills

In non-abelian gauge theory we promote V to a Hermitian matrix, $V = V^a T^a$. Here T^a are a set of generators normalized to $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$. In WZ gauge V contains the gauge field $A_\mu = A^a_\mu T^a$, an adjoint spinor field $\lambda_\alpha = \lambda^a_\alpha T^a$, and an adjoint scalar auxiliary field $D = D^a T^a$. You can couple V to a collection of chiral multiplets Φ_i in (say) the fundamental representation. The general supersymmetric and gauge-invariant action is

$$S = \int d^4x d^4\theta \sum_i \Phi_i^\dagger e^{gV} \Phi_i + \left[\int d^4x d^2\theta \left(\frac{1}{2} \mathrm{Tr} W^\alpha W_\alpha + W(\Phi) \right) + \text{c.c.} \right] .$$

Here g is the gauge coupling and the superpotential W is a gauge-invariant holomorphic function of the Φ_i's. Gauge transformations act on these fields via

$$\Phi_i \to e^{-ig\Lambda} \Phi_i \qquad e^{gV} \to e^{-ig\Lambda^\dagger} e^{gV} e^{ig\Lambda}$$

where Λ is an adjoint chiral multiplet. Expanding the action in components you find (in WZ gauge)

$$S = \int d^4x \; -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} - (\mathcal{D}_\mu \phi_i)^\dagger \mathcal{D}^\mu \phi_i + \mathcal{V}(\phi_i, \phi_i^\dagger) + \text{fermions}$$

where the potential energy is a sum of F-term and D-term contributions.

$$\begin{aligned} \mathcal{V} &= \sum_i F_i^\dagger F_i + \frac{1}{2} \sum_a (D^a)^2 \\ &= \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{1}{2} g^2 \sum_a \left(\sum_i \phi_i^\dagger T^a \phi_i \right)^2 \end{aligned}$$