

1. Vacuum alignment in QCD

Strong interactions are supposed to generate a non-zero expectation value that spontaneously breaks $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$. The space of vacua can be parametrized by a unitary matrix $U = e^{i\lambda^a T^a}$ that characterizes the expectation value

$$\langle \psi \bar{\psi} \rangle = \mu^3 e^{-i\lambda^a T^a \gamma^5}.$$

Here μ is a constant with dimensions of mass. The low energy effective Lagrangian for the resulting Goldstone bosons is

$$\mathcal{L} = \frac{1}{4} f^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U)$$

where f is another constant with dimensions of mass.

- (i) Consider adding a quark mass term $\mathcal{L}_{\text{mass}} = -\bar{\psi} M \psi$ to the underlying strong interaction Lagrangian. Argue that for small quark masses the explicit breaking due to the mass term can be taken into account by modifying the effective Lagrangian to read

$$\frac{1}{4} f^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + 2\mu^3 \text{Tr}(M(U + U^\dagger)).$$

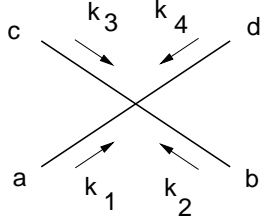
- (ii) Identify the ground state of the resulting theory. Compute the matrix of would-be Goldstone boson masses by expanding the action to quadratic order in the fields π^a , where π^a is defined by $U = e^{i\pi^a T^a/f}$ with $\text{Tr} T^a T^b = 2\delta^{ab}$.
- (iii) Use your results to predict the η mass in terms of $m_{\pi^\pm}^2$, $m_{\pi^0}^2$, $m_{K^\pm}^2$, $m_{K^0}^2$, $m_{\bar{K}^0}^2$. How does your prediction compare to the data? (You can ignore small isospin breaking effects and set $m_u = m_d$.)

2. $\pi - \pi$ scattering

If you take the pion effective Lagrangian

$$\mathcal{L} = \frac{1}{4} f^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + 2\mu^3 \text{Tr}(M(U + U^\dagger))$$

and expand it to fourth order in the pion fields you find interaction terms that describe low-energy $\pi - \pi$ scattering. Conventions: $U = e^{i\pi^a \sigma^a / f}$ is an $SU(2)$ matrix where σ^a are Pauli matrices and $f = 93 \text{ MeV}$. I'm working in a Cartesian basis where $a = 1, 2, 3$. For simplicity I'll set $m_u = m_d$ so that isospin is an exact symmetry. The resulting 4-pion vertex is (sorry)



$$-\frac{i}{3f^2} \left[\delta^{ab} \delta^{cd} \left((k_1 + k_2) \cdot (k_3 + k_4) - 2k_1 \cdot k_2 - 2k_3 \cdot k_4 - m_\pi^2 \right) \right. \\ \left. + \delta^{ac} \delta^{bd} \left((k_1 + k_3) \cdot (k_2 + k_4) - 2k_1 \cdot k_3 - 2k_2 \cdot k_4 - m_\pi^2 \right) \right. \\ \left. + \delta^{ad} \delta^{bc} \left((k_1 + k_4) \cdot (k_2 + k_3) - 2k_1 \cdot k_4 - 2k_2 \cdot k_3 - m_\pi^2 \right) \right]$$

Note that all momenta are directed inwards in the vertex.

- (i) Compute the scattering amplitude for $\pi^a(p_1) \pi^b(p_2) \rightarrow \pi^c(p_3) \pi^d(p_4)$. Here a, b, c, d are isospin labels and p_1, p_2, p_3, p_4 are external momenta.
- (ii) Since we have two pions the initial state could have total isospin $I = 0, 1, 2$. Extract the scattering amplitude in the various isospin channels by putting in initial isospin wavefunctions proportional to

$$I = 0 : \delta^{ab} \quad I = 1 : (\text{antisymmetric})^{ab} \quad I = 2 : (\text{symmetric traceless})^{ab}$$

- (iii) Evaluate the amplitudes in the various channels “at threshold” (meaning in the limit where the pions have vanishing spatial momentum).
- (iv) Threshold scattering amplitudes are usually expressed in terms of “scattering lengths” defined (for s -wave scattering) by $a = -\mathcal{M}/32\pi m_\pi$.¹² For $I = 0, 2$ the experimental values and statistical errors are (Brookhaven E865 collaboration, arXiv:hep-ex/0301040)

$$a_{I=0} = (0.216 \pm 0.013)m_\pi^{-1} \quad a_{I=2} = (-0.0454 \pm 0.0031)m_\pi^{-1}$$

How well did you do?

¹This is in the convention where the sum of Feynman diagrams gives $-i\mathcal{M}$.

²This seems to be the standard definition, but it never quite made sense to me. I would have thought the scattering length should be defined by $a = -\mathcal{M}/16\pi m_\pi$ so that the cross section comes out to be $\sigma = 4\pi a^2$. If anyone understands this please let me know.