Physics G8069
homework #8
due Mon. Nov. 13

1. **Mass renormalization in ϕ^4 theory**

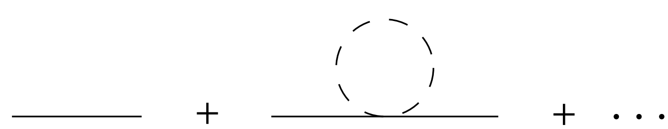
Let's study renormalization of the mass parameter in ϕ^4 theory. As in the notes we consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

with a cutoff on the Euclidean loop momentum Λ, and

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m'^2\phi'^2 - \frac{1}{4!}\lambda'\phi'^4$$

with a cutoff $\Lambda' = \Lambda - \delta\Lambda$. The idea is to match the tree-level propagator in the primed theory to the corresponding quantity in the unprimed theory, namely the sum of diagrams



(i) In the primed theory the propagator is

$$\frac{i}{p^2 - m'^2}\,.$$

Set $m'^2 = m^2 + \delta m^2$ where $\delta m^2 = -\frac{dm^2}{d\Lambda}\delta\Lambda$. Expand the propagator to first order in $\delta\Lambda$.

(ii) Match your answer to the corresponding calculation in the unprimed theory. You can stop at a single χ loop, and for simplicity you can assume $m \ll \Lambda$. You should obtain a trivial differential equation for the mass parameter and solve it to find $m^2(\Lambda)$.

2. **Pion decay**

(i) Derive the Noether currents $j_L^{\mu\,a}$, $j_R^{\mu\,a}$ associated with the $SU(2)_L \times SU(2)_R$ symmetry

$$\delta\psi_L = -\frac{i}{2}\lambda_L^a \sigma^a \psi_L \qquad \delta\psi_R = -\frac{i}{2}\lambda_R^a \sigma^a \psi_R$$

of QCD with two flavors of massless quarks. We'll mostly be interested in the vector and axial-vector linear combinations $j_V^{\mu\,a} = j_L^{\mu\,a} + j_R^{\mu\,a}$, $j_A^{\mu\,a} = -j_L^{\mu\,a} + j_R^{\mu\,a}$.

(ii) Repeat part (i) for the $SU(2)$ non-linear σ-model

$$\mathcal{L} = \frac{1}{4} f^2 \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)$$

where the symmetry is

$$\delta U = -\frac{i}{2}\lambda_L^a \sigma^a U + U \frac{i}{2}\lambda_R^a \sigma^a \,.$$

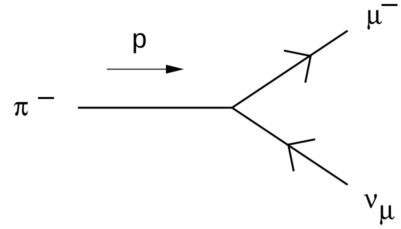
You only need to work out the symmetry currents to first order in the pion fields π^a, where $U = e^{i\pi^a\sigma^a/f}$.

(iii) The weak interaction responsible for the decay $\pi^- \to \mu^- \bar{\nu}_\mu$ is

$$\mathcal{L}_{\mathrm{weak}} = -\frac{1}{\sqrt{2}} G_F \cos\theta_C \, \bar{\mu}\gamma^\lambda (1-\gamma^5)\nu_\mu \, \bar{u}\gamma_\lambda (1-\gamma^5) d + h.c.$$

Here $\theta_C \approx 13°$ is the 'Cabibbo angle.' Suppose we can identify the symmetry currents worked out in parts (i) and (ii). Use this to rewrite $\mathcal{L}_{\mathrm{weak}}$ in terms of the fields μ, ν_μ, π^a, again working to first order in the pion fields.

(iv) If I did it right this leads to a vertex



$$G_F \cos\theta_C f \gamma^\mu (1-\gamma^5) p_\mu$$

Calculate the pion lifetime in terms of G_F, θ_C, f, m_π, m_μ. Use the observed lifetime $\tau_\pi = 2.6 \times 10^{-8}$ sec to estimate f.

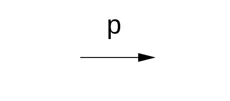
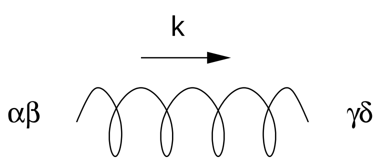
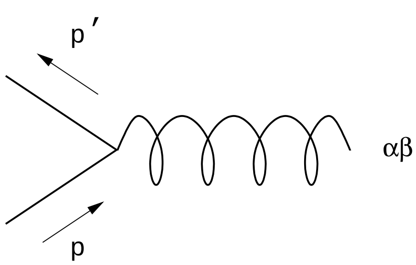
(v) The decay $\pi^- \to e^- \bar{\nu}_e$ only differs by replacing $\mu \to e$, $\nu_\mu \to \nu_e$. Predict the branching ratio

$$\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)}$$

How well did you do?

3. **Unitarity violation in quantum gravity**

Consider two distinct types of massless scalar particles A and B which only interact gravitationally. The Feynman rules are

	scalar propagator	$\dfrac{i}{p^2}$
	graviton propagator	$\dfrac{i16\pi G_N}{k^2}\left(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta}\right)$
	scalar – graviton vertex	$\frac{i}{2}\left(p_\alpha p'_\beta + p_\beta p'_\alpha - g_{\alpha\beta}\, p \cdot p'\right)$

Here G_N is Newton's constant and $g_{\alpha\beta} = \mathrm{diag}(+---)$ is the Minkowski metric. The vertices and propagators are the same for A and B.

(i) Compute the tree-level amplitude and center-of-mass differential cross section for the process $AA \to BB$.

(ii) The partial-wave expansion of the scattering amplitude is

$$f(\theta) = \frac{1}{i\sqrt{s}} \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, S_l(E)$$

where P_l is a Legendre polynomial. This is related to the center-of-mass differential cross section by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm cm} = |f(\theta)|^2 \, .$$

Compute the partial-wave S-matrix elements $S_l(E)$. For which values of l are they non-zero?

(iii) At what center-of-mass energy is the unitarity bound $|S_l(E)| \leq 1$ violated?