Physics G8069
homework #4
due Mon. Oct. 9

1. $O(N)$ **linear σ-model**

   Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2}\mu^2 |\phi|^2 - \frac{1}{4}\lambda |\phi|^4$$

   Here ϕ is a vector containing N scalar fields. Note that $\mathcal{L}$ is invariant under rotations $\phi \to R\phi$ where $R \in SO(N)$.

   (i) Find the conserved currents associated with this symmetry.

   (ii) When $\mu^2 < 0$ the $SO(N)$ symmetry is spontaneously broken. In this case identify

   - the space of vacua
   - the unbroken symmetry group
   - the spectrum of particle masses

2. $O(4)$ **linear σ-model**

   Specialize to $N = 4$ and define $\Sigma = \phi_4 \mathbb{1} + \sum_{k=1}^{3} i\phi_k \tau_k$ where τ_k are Pauli matrices.

   (i) Show that $\det \Sigma = |\phi|^2$ and $\Sigma^* = \tau_2 \Sigma \tau_2$.

   (ii) Rewrite $\mathcal{L}$ in terms of Σ.

   (iii) In place of $SO(4)$ transformations on ϕ we now have $SU(2)_L \times SU(2)_R$ transformations on Σ. These transformations act by $\Sigma \to L\Sigma R^\dagger$ where $L, R \in SU(2)$. Show that these transformations leave $\det \Sigma$ invariant and preserve the property $\Sigma^* = \tau_2 \Sigma \tau_2$.

   (iv) Show that one can set $\Sigma = \sigma U$ where $\sigma > 0$ and $U \in SU(2)$.

(v) Rewrite the Lagrangian in terms of σ and U. Take $\mu^2 < 0$ so the $SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken and, in terms of the fields σ and U, identify

- the space of vacua
- the unbroken symmetry group
- the spectrum of particle masses

(vi) Write down the low energy effective action for the Goldstone bosons.

3. **Vacuum alignment in the σ-model**

Suppose we add an explicit symmetry-breaking perturbation to our $O(4)$ linear σ-model Lagrangian.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2}\mu^2 |\phi|^2 - \frac{1}{4}\lambda |\phi|^4 + \mathbf{a} \cdot \phi$$

Here $\mu^2 < 0$ and $\mathbf{a}$ is a constant vector; for simplicity you can take it to point in the ϕ_4 direction. What is the unbroken symmetry group? Identify the (unique) vacuum state and expand about it by setting

$$\Sigma = (f + \rho) e^{i\pi \cdot \tau / f}$$

Here f is a constant and ρ and π are fields with $\langle \rho \rangle = \langle \pi \rangle = 0$. Identify the spectrum of particle masses.

4. $SU(N)$ **nonlinear σ-model**

Consider the Lagrangian $\mathcal{L} = \frac{1}{4} f^2 \operatorname{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)$ where f is a constant with units of $(\text{mass})^2$ and $U \in SU(N)$. The Lagrangian is invariant under $U \to L U R^\dagger$ where $L, R \in SU(N)$. Identify

- the space of vacua
- the unbroken symmetry group
- the spectrum of particle masses