References: Supersymmetric quantum mechanics was introduced (or at least popularized) by Witten, *Nucl. Phys.* **B185** (1981) 513 section 6. I learned about the superfield formulation from a book by Misra, *Introduction to supersymmetry and supergravity*, although there might be a better reference.

1.1 Why supersymmetry?

Supersymmetry is an extension of Poincaré (= Lorentz plus translation) invariance which has the surprising feature of mixing bosons and fermions. Why study it?

1. With some reasonable assumptions, supersymmetry is the only possible extension of Poincaré invariance to a larger spacetime symmetry (the Haag - Sohnius - Lopuszanski theorem).

2. Remarkably, supersymmetry is the leading candidate for physics beyond the standard model. Susy theories can explain the weak hierarchy $m_{\text{weak}} \ll m_{\text{Planck}}$, they yield gauge coupling unification at the GUT scale, and they produce natural dark matter candidates.

3. Supersymmetry plays a crucial role in constructing well-behaved string theories. My guess is that it's necessary for a consistent theory of quantum gravity.

4. Supersymmetry is a powerful tool for obtaining exact results in interesting strongly-coupled quantum field theories.

'nuff said.

1.2 Supersymmetric quantum mechanics: Hamiltonian approach

The simplest example of a supersymmetric system is actually a problem in non-relativistic quantum mechanics, a spin-1/2 particle moving on a line in a magnetic field. The Hamiltonian is

$$H = \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right)\mathbb{1} + \frac{1}{2}B(x)\sigma^3 \tag{1}$$

acting on a two-component wavefunction. Here $V(x)$ is the potential, $B(x)$ is the magnetic field which points in the z direction, and σ^3 is a Pauli matrix. This Hamiltonian was introduced by Witten, who pointed out that it's supersymmetric provided you make the specific choices

$$V(x) = \frac{1}{2}\left(\frac{dW}{dx}\right)^2 \qquad B(x) = \frac{d^2W}{dx^2}\,.$$

Here $W(x)$ is an arbitrary function known as the superpotential.

What does it mean for H to be supersymmetric? Define the operators

$$Q = \left(-i\frac{d}{dx} - i\frac{dW}{dx}\right)\sigma^+ \qquad Q^\dagger = \left(-i\frac{d}{dx} + i\frac{dW}{dx}\right)\sigma^- \tag{2}$$

where $\sigma^+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\sigma^- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ are spin raising and lowering operators. Q and $Q^\dagger$ are known as supercharges. A little calculation shows that they obey the commutation / anticommutation relations

$$\begin{aligned} &Q^2 = (Q^\dagger)^2 = 0 \\ &\{Q, Q^\dagger\} = 2H \\ &[Q, H] = [Q^\dagger, H] = 0 \end{aligned}$$

Relations of this form characterize a supersymmetry algebra.

Supersymmetry has remarkable consequences. Consider a normalized energy eigenstate

$$H|E\rangle = E|E\rangle \qquad \langle E|E\rangle = 1\,.$$

The supersymmetry algebra implies

$$E = \langle E|H|E\rangle = \langle E|\frac{1}{2}(QQ^\dagger + Q^\dagger Q)|E\rangle = \frac{1}{2}\|Q^\dagger|E\rangle\|^2 + \frac{1}{2}\|Q|E\rangle\|^2 \tag{3}$$

This expresses the energy eigenvalue as a sum of non-negative terms, so in a supersymmetric theory

$$\begin{aligned} &E \geq 0 \\ &E = 0 \quad \text{iff} \quad Q|E\rangle = Q^\dagger|E\rangle = 0 \end{aligned}$$

Supersymmetry even lets us find the ground state wavefunction explicitly. Let's assume that $W(x) \to +\infty$ as $x \to \pm\infty$.[1] Then given

$$Q = \begin{pmatrix} 0 & -i\partial_x - iW' \\ 0 & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ -i\partial_x + iW' & 0 \end{pmatrix}$$

it's easy to see that

$$Q\Psi = Q^\dagger\Psi = 0 \quad \Rightarrow \quad \Psi = \begin{pmatrix} Ae^{W(x)} \\ Be^{-W(x)} \end{pmatrix}$$

where A and B are constants. To get a normalizeable state we must set $A = 0$ – this is where the assumption about the behavior of W comes in – so the ground state wavefunction is

$$\Psi_0 = \begin{pmatrix} 0 \\ Be^{-W(x)} \end{pmatrix}$$

So much for the ground state. What about excited states? They all come in degenerate spin-up/spin-down pairs, related by the action of Q, $Q^\dagger$. To see this it's helpful to note that $[H, \sigma^3] = 0$, so that energy eigenstates can be assigned a definite spin in the z direction.

1.3 Lagrangian formulation

In this section we'll change perspective and regard the quantum mechanics problem as a supersymmetric field theory in 0+1 dimensions. Let me emphasize that this is purely a change in notation! Physically we're studying exactly the same system.

[1] We'll consider other possibilities later.

Let's postulate a field theory and show that upon quantization it reproduces the Hamiltonian (1). First, the field content. Introduce

$x(t)$	real scalar field
$\psi(t)$	complex Grassmann field
$\bar{\psi}(t)$	complex conjugate of ψ

These fields are functions of the spacetime coordinates, which in 0+1 means just the time coordinate t. Grassmann numbers behave like ordinary numbers except that multiplication anticommutes. In our case this means

$$\psi^2 = \bar{\psi}^2 = 0 \qquad\qquad \psi\bar{\psi} = -\bar{\psi}\psi\,.$$

We adopt the rule that complex conjugation reverses the order of Grassmann numbers, just like Hermitian conjugation of operators, so that if ψ_1 and ψ_2 are Grassmann numbers

$$(\psi_1\psi_2)^* = \bar{\psi}_2\bar{\psi}_1\,.$$

Next we postulate an action: $S = \int dt\, L$ where

$$L = \frac{1}{2}\dot{x}^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}\left(\frac{dW}{dx}\right)^2 + \bar{\psi}\psi\frac{d^2W}{dx^2} \tag{4}$$

Note that the Lagrangian is real. The first term is the standard kinetic term for a scalar field in 0+1 dimensions. The second term can be thought of as the Dirac Lagrangian in 0+1. The 3rd term is a potential energy for the scalar field, while depending on the exact form of W the last term could give rise to fermion mass terms, Yukawa couplings, etc.

To show that this field theory is equivalent to Witten's quantum mechanics we first switch to a Hamiltonian description. I'm going to cut a few corners here.[2] We define the conjugate momenta

$p = \dot{x}$	momentum conjugate to x
$\pi = i\bar{\psi}$	momentum conjugate to ψ

[2]For a discussion of classical mechanics with Grassmann variables see chapter 6 in Henneaux and Teitelboim, *Quantization of gauge systems*. Also for a nice treatment of (bosonic) first-order Lagrangians see Faddeev and Jackiw, Phys.Rev.Lett.60:1692,1988.

The Hamiltonian is

$$\begin{aligned} H &= p\dot{x} + \frac{1}{2}(\pi\dot{\psi} + \dot{\bar{\psi}}\pi) - L \\ &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 - \bar{\psi}\psi\frac{d^2W}{dx^2}\,. \end{aligned}$$

To quantize we adopt the equal-time (anti-) commutation relations

$$\begin{aligned} &[\hat{x}, \hat{p}] = i \\ &\{\hat{\psi}, \hat{\pi}\} = i \quad \text{or equivalently} \quad \{\hat{\bar{\psi}}, \hat{\psi}\} = 1 \end{aligned} \tag{5}$$

We have to choose an operator ordering for our quantum Hamiltonian. A reasonable ordering – as we'll see, the one required by supersymmetry – is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 - \frac{1}{2}(\hat{\bar{\psi}}\hat{\psi} - \hat{\psi}\hat{\bar{\psi}})\frac{d^2W}{dx^2}\,. \tag{6}$$

The real justification for this quantization procedure is that (up to operator ordering) the Heisenberg equations of motion for this Hamiltonian agree with the Euler-Lagrange equations of motion for (4).

The next step is to find linear operators which realize the (anti-) commutation relations (5). A canonical choice[3] is

$$\hat{p} = -i\frac{\partial}{\partial x} \qquad \hat{\psi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{\bar{\psi}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

With this choice one can readily show that

$$\hat{H} = \left(-\partial_x^2 + \frac{1}{2}(W')^2\right)\mathbb{1} + \frac{1}{2}W''\sigma^3\,.$$

This is exactly the Hamiltonian (1) that Witten postulated.

What about supersymmetry? Consider the following transformation of the fields:

$$\begin{aligned} \delta x &= \xi\psi + \bar{\psi}\bar{\xi} \\ \delta\psi &= -i\bar{\xi}(\dot{x} + iW') \\ \delta\bar{\psi} &= i\xi(\dot{x} - iW') \end{aligned} \tag{7}$$

[3]unique up to unitary transformations

Here ξ is a constant complex Grassmann parameter. These transformations mix bose and fermi fields – a hallmark of supersymmetry – and with a bit of effort one can check that the transformation leaves the action invariant (the Lagrangian changes by a total time derivative).

For every symmetry there should be a conservation law. Rather than give a general derivation of the fermionic version of Noether's theorem, I'll specialize to the problem at hand and get the conservation law by promoting the parameters appearing in the supersymmetry transformations (7) to time-dependent quantities.

$$\xi \mapsto \xi(t) \qquad \bar{\xi} \mapsto \bar{\xi}(t)$$

When ξ and $\bar{\xi}$ are allowed to depend on time we no longer have a symmetry transformation. With a bit of calculation one can show that the change in the Lagrangian is

$$\begin{aligned} \delta L &= -\frac{d}{dt}\left(\frac{1}{2}(\dot{x}+iW')\bar{\xi}\bar{\psi} - \frac{1}{2}(\dot{x}-iW')\xi\psi - iW'(\xi\psi + \bar{\xi}\bar{\psi})\right) \\ &\quad +\dot{\xi}(\dot{x}-iW')\psi - \dot{\bar{\xi}}(\dot{x}+iW')\bar{\psi} \end{aligned} \tag{8}$$

What can we conclude from this?

- If ξ and $\bar{\xi}$ are constant then, as claimed, the Lagrangian changes by a total time derivative.
- Suppose the fields satisfy their equations of motion. Then the action should be invariant under arbitrary variations of the fields,

  $$\delta S = \int dt\, \delta L = 0\,.$$

  (In general one has to impose boundary conditions to make the surface terms in δS vanish.) Now consider the variation (7) where the parameters ξ, $\bar{\xi}$ vanish as $t \to \pm\infty$ but are otherwise arbitrary. One can compute δS for this variation, by integrating the first two terms in (8) by parts, and one sees that the equations of motion (requiring $\delta S = 0$) imply that the quantities

  $$\begin{aligned} Q &= (\dot{x}-iW')\psi \\ \bar{Q} &= (\dot{x}+iW')\bar{\psi} \end{aligned} \tag{9}$$

satisfy $\frac{dQ}{dt} = \frac{d\bar{Q}}{dt} = 0$. These are the conserved supercharges associated with supersymmetry.

A few comments are in order.

1. It's easy to see that upon quantization the supercharges (9) agree with the operators (2) we wrote down previously.
2. The fact that these operators satisfy $\{Q, Q^\dagger\} = 2H$ means we chose the correct operator ordering in our Hamiltonian (6).
3. As one might expect, the supercharges generate supersymmetry transformations. Introduce the bosonic Hermitian linear combination $G = \xi Q - \bar{\xi} Q^\dagger$. One can check that (7) is equivalent to

$$\begin{aligned}
\delta x &= i[G, x] \\
\delta \psi &= i[G, \psi] \\
\delta \bar{\psi} &= i[G, \bar{\psi}]
\end{aligned}$$

1.4 Auxiliary fields

Symmetries are useful because they constrain the form of the Lagrangian. We're used to defining a field theory by specifying the field content and the symmetries, then working out the most general Lagrangian compatible with the symmetries. For supersymmetry this program seems to crash. The susy transformations we've worked out

$$\begin{aligned}
\delta x &= \xi \psi + \bar{\psi} \bar{\xi} \\
\delta \psi &= -i \bar{\xi} (\dot{x} + i W') \\
\delta \bar{\psi} &= i \xi (\dot{x} - i W')
\end{aligned}$$

depend explicitly on the superpotential, so they depend on the very Lagrangian we'd like to construct. There's nothing wrong with this, but it makes it very hard to exploit the power of supersymmetry.

Fortunately there's a cure for this sort of problem, which comes up again and again in formulating susy theories. Introduce an additional real bosonic field $d(t)$ and consider the Lagrangian

$$L_{\text{aux}} = \frac{1}{2}\dot{x}^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) + \frac{1}{2}d^2 - dW' + \bar{\psi}\psi W'' \,. \tag{10}$$

Note that the Lagrangian doesn't involve any time derivatives of d, so d isn't a propagating degree of freedom. It's known as an auxiliary field. This theory is completely equivalent to (4). You can see this both at the classical and quantum levels.

Classical argument

Varying the action, the equation of motion for d is just $\frac{\partial L}{\partial d} = 0$ which fixes $d = W'(x)$. Again this shows that d isn't an independent degree of freedom: it's a constrained variable, fixed by its equation of motion in terms of x. Substituting $d = W'$ back into (10) reproduces (4).

Quantum argument

d appears quadratically in the Lagrangian. Performing the Gaussian path integral over d gives

$$\int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}d\, e^{i\int dt L_{\text{aux}}} = \text{const.} \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{i\int dt L_{\text{aux}}|_{d=W'}} \,.$$

The overall constant can be absorbed into the normalization of the path integral.

How does supersymmetry look with auxiliary fields? Repeating the arguments towards the end of the last section you can show that the action is invariant under the supersymmetry transformation

$$\begin{aligned} \delta x &= \xi\psi + \bar{\psi}\bar{\xi} \\ \delta\psi &= -i\bar{\xi}(\dot{x} + id) \\ \delta\bar{\psi} &= i\xi(\dot{x} - id) \\ \delta d &= -i(\xi\dot{\psi} + \bar{\xi}\dot{\bar{\psi}}) \,. \end{aligned} \tag{11}$$

The corresponding supercharges

$$\begin{aligned} Q &= (\dot{x} - id)\psi \\ \bar{Q} &= (\dot{x} + id)\bar{\psi} \end{aligned}$$

are conserved, $\frac{dQ}{dt} = \frac{d\bar{Q}}{dt} = 0$, but showing that these supercharges generate the right susy transformations seems non-trivial.

We've achieved our goal, since the supersymmetry transformations (11) make no reference to any Lagrangian. You might be left wondering how the transformations (11) and (7) are related. On the homework you'll show that they're equivalent for field configurations that satisfy the equations of motion.

1.5 Superfields

We're getting there... the only problem with the susy transformations (11) is that it's hardly obvious that they're a symmetry of the Lagrangian (10). We're going to introduce some notation with the goal of making supersymmetry manifest.

To this end we introduce "superspace," a space with one real bosonic coordinate t and a complex conjugate pair of Grassmann coordinates θ, $\bar{\theta}$.

$$\text{superspace} = \{(t, \theta, \bar{\theta})\}$$

We also introduce the notion of a "superfield", a function F on superspace with definite transformation properties under supersymmetry.

$$F = F(t, \theta, \bar{\theta})$$

We'll specify the susy transformation properties shortly.

This seems very abstract, but if you recall the rules for Grassmann variables

$$\theta^2 = \bar{\theta}^2 = 0 \qquad \theta\bar{\theta} = -\bar{\theta}\theta$$

you realize that superfields have very simple Taylor series expansions in powers of the Grassmann coordinates:

$$F(t, \theta, \bar{\theta}) = x(t) + \theta\psi(t) + \bar{\psi}(t)\bar{\theta} + \theta\bar{\theta}d(t)\,.$$

This is known as the component expansion of the superfield. Concentrating on the simple case where F is a real bosonic function on superspace we see that $x(t)$ is a real bosonic field, $\psi(t)$ is a complex Grassmann field, $\bar{\psi}(t)$ is the complex conjugate of $\psi(t)$, and $d(t)$ is a real bosonic field.

How should supersymmetry act on a superfield? Here we'll postulate that the individual components of the superfield have the susy transformation properties given in (11).

$$\delta F = \delta x + \theta \delta \psi + (\delta \bar{\psi})\bar{\theta} + \theta \bar{\theta} \delta d$$

This is part of our definition of a superfield. With a little algebra one can show that

$$\delta F = \xi Q_\partial F + \bar{\xi} \bar{Q}_\partial F \tag{12}$$

where we've defined some differential operators on superspace[4]

$$Q_\partial = \partial_\theta + i\bar{\theta}\partial_t \qquad \bar{Q}_\partial = \partial_{\bar{\theta}} + i\theta\partial_t \,.$$

The notation is a little confusing. The differential operators Q_∂, $\bar{Q}_\partial$ are not to be confused with the conserved supercharges Q, $\bar{Q}$. Also $\bar{Q}_\partial$ isn't related to Q_∂ by complex conjugation. Rather one has the somewhat counterintuitive identity $\bar{Q}_\partial F = -(Q_\partial F)^*$; note that this makes the right hand side of (12) real.

This leads to a very nice geometrical interpretation of supersymmetry as a type of translation in superspace. The best way to understand this is by analogy with ordinary time translation, as generated by some Hamiltonian H.

$$i[\epsilon H, F] = \epsilon \partial_t F \quad \Leftrightarrow \quad H \text{ generates time translation } t \rightarrow t + \epsilon$$

The analogous statement for supersymmetry is

$$i[\xi Q - \bar{\xi} Q^\dagger, F] = \xi Q_\partial F + \bar{\xi} \bar{Q}_\partial F \quad \Leftrightarrow \quad \text{supercharges generate translations in superspace}$$

What is the translation? To find out we just have to let the differential operators act on the superspace coordinates.

$$\delta(t, \theta, \bar{\theta}) = (\xi Q_\partial + \bar{\xi} \bar{Q}_\partial)(t, \theta, \bar{\theta}) = (i\xi\bar{\theta} + i\bar{\xi}\theta, \xi, \bar{\xi})$$

[4] Grassmann derivatives are defined by $\partial_\theta \theta = \partial_{\bar{\theta}} \bar{\theta} = 1$, $\partial_\theta \bar{\theta} = \partial_{\bar{\theta}} \theta = 0$.

That is, supersymmetry acts as a translation

$$\theta \to \theta + \xi \qquad \bar{\theta} \to \bar{\theta} + \bar{\xi} \qquad t \to t + i(\xi\bar{\theta} + \bar{\xi}\theta)$$

on the superspace coordinates. Again you can see that supersymmetry is an extension of the Poincaré group to include translations in the Grassmann directions.

1.6 Constructing supersymmetric actions

Now let's see if we can use superfields to rewrite the action in a way that makes supersymmetry manifest.

In order to write supersymmetric kinetic terms we need a derivative operator that is compatible with supersymmetry. The construction is a little subtle.[5] Define the differential operators

$$D = \partial_\theta - i\bar{\theta}\partial_t \qquad \bar{D} = \partial_{\bar{\theta}} - i\theta\partial_t \,.$$

These are known as supercovariant derivatives. They look a lot like the operators Q_∂, $\bar{Q}_\partial$ except that the ∂_t terms have opposite sign. A little calculation reveals that the D's and Q_∂'s all anticommute,

$$\{D, Q_\partial\} = \{D, \bar{Q}_\partial\} = \{\bar{D}, Q_\partial\} = \{\bar{D}, \bar{Q}_\partial\} = 0 \,.$$

Why is this important? Let's work out the susy variation of DF.

$$\delta_{susy} DF = D\delta_{susy} F = D(\xi Q_\partial + \bar{\xi}\bar{Q}_\partial)F = (\xi Q_\partial + \bar{\xi}\bar{Q}_\partial)DF$$

(To see the first equality imagine writing everything in terms of component fields. The last equality relies on the anticommutation relations.) So DF has the same susy transformation rule as F itself: the supercovariant derivative maps superfields to superfields. Likewise for $\bar{D}$.

Now, can we write a supersymmetric action? It's easy to see that the product of two superfields has the right transformation properties to be another superfield. So consider the superspace Lagrangian

$$L_{\text{super}} = -\frac{1}{2}\bar{D}F DF - W(F)$$

[5] It has a geometric interpretation in superspace given on p. 26 of Wess & Bagger.

where the superpotential W is an arbitrary function of F. L is a real[6] bosonic superfield, and it involves at most two time derivatives, so this seems like a promising starting point. Can we make an invariant action out of L? Recall that the susy variation of the top component of a superfield is a total derivative:

$$
\begin{aligned}
F &= \cdots + \theta\bar{\theta} d \\
\delta_{susy} d &= -i \frac{d}{dt}(\xi\psi + \bar{\xi}\bar{\psi})
\end{aligned}
$$

This is true for any superfield, so the action

$$
S = \int dt \, (L_{\text{super}})_{\theta\bar{\theta} \text{ component}}
$$

is invariant under supersymmetry. This is usually written in a fancier way. Grassmann integration is defined by

$$
\begin{aligned}
&\int d\theta = 0 \qquad \int d\theta \, \theta = 1 \qquad \int d\bar{\theta} = 0 \qquad \int d\bar{\theta} \, \bar{\theta} = 1 \\
&d^2\theta \equiv d\bar{\theta} d\theta \quad \text{so} \int d^2\theta \, \theta\bar{\theta} = 1
\end{aligned}
$$

This means for any superfield

$$
\int d^2\theta \, F = (F)_{\theta\bar{\theta} \text{ component}} .
$$

In particular we can write the manifestly supersymmetric action

$$
S = \int dt d^2\theta \left(-\frac{1}{2} \bar{D} F D F - W(F) \right) .
$$

I leave it as a homework exercise to show that this is equivalent to the component action (10).

[6] One has the somewhat counterintuitive identity $(DF)^* = -\bar{D}F$.