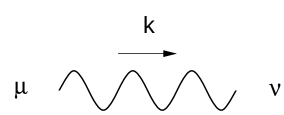
Physics G8069
homework #9
due Mon. Nov. 20

1. **Interaction potential in QED**

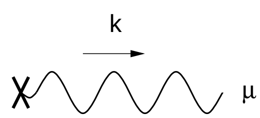
Consider coupling the electromagnetic field to a conserved external current $J^\mu(x)$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \,.$$

The Feynman rules for this theory are



$$\frac{-ig_{\mu\nu}}{k^2}$$



$$-iJ_\mu(k)$$

where $J_\mu(k) = \int d^4x\, e^{ik\cdot x} J_\mu(x)$. These rules are set up so the sum of connected Feynman diagrams gives $-i\int d^4x\, \mathcal{H}_{\rm int}$ where $\mathcal{H}_{\rm int}$ is the energy density due to interactions.

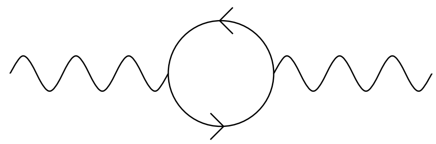
(i) Introduce two point charges Q_1, Q_2 at positions $\mathbf{x}_1$, $\mathbf{x}_2$ by setting

$$\begin{aligned} &J^0(x) = eQ_1\delta^3(\mathbf{x}-\mathbf{x}_1) + eQ_2\delta^3(\mathbf{x}-\mathbf{x}_2) \\ &J^i = 0 \qquad i = 1,2,3 \end{aligned}$$

We're measuring the charges in units of $e = \sqrt{4\pi\alpha}$. Compute the interaction energy by evaluating the diagram

You should integrate over the photon momentum. Do you recover the usual Coulomb potential?

(ii) The photon propagator receives corrections from a virtual $e^+ - e^-$ loop via the diagram



As we showed in class this diagram equals

$$C\frac{g_{\mu\nu}}{k^4} + \frac{4e^2}{3k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4} + \mathcal{O}(k^0) \tag{1}$$

Here we're neglecting the electron mass and including the propagators on the external lines. We've done a Taylor series expansion in powers of the photon momentum k. Also p is the loop momentum and C is a (quadratically divergent) constant which we can ignore – it would vanish if we cut off the loop integral in a gauge invariant way.

Now consider the potential between two point charges when these corrections to the photon propagator are taken into account. Suppose the charges are widely separated, so that the potential is dominated by the behavior of the photon propagator at small values of k. That is, suppose we only need to pay attention to the $\mathcal{O}(1/k^2)$ terms in (1). Introduce a cutoff on the Euclidean loop momentum, $|p_E| < \Lambda$, and allow the electric charge to depend on Λ, $e^2 \rightarrow e^2(\Lambda)$. Show that up to $\mathcal{O}(e^4)$ the tree plus one-loop potential is independent of Λ provided

$$\frac{de^2}{d\Lambda} = \frac{e^4}{6\pi^2\Lambda}$$

or equivalently

$$\frac{1}{e^2(\Lambda)} = \frac{1}{e^2(\mu)} - \frac{1}{6\pi^2}\log\frac{\Lambda}{\mu}$$

Here μ is an arbitrary renormalization scale. Hints: since the external current is conserved, $k_\mu J^\mu(k) = 0$, you can drop corrections to the photon propagator proportional to k^μ. Also it helps to note that $\frac{de^2}{d\Lambda}$ is $\mathcal{O}(e^4)$.

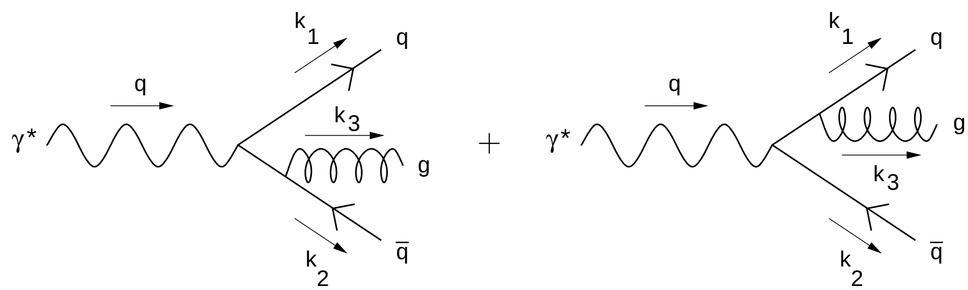
(iii) Similar to the scattering problem from homework #7: suppose you were interested in the potential between two unit charges separated by a distance r. Working in terms of the renormalized coupling, the tree diagram gives a potential $e^2(\mu)/4\pi r$. How should you choose the renormalization scale to make the loop corrections to this as small as possible? Hint: think about which photon momentum makes the dominant contribution to the potential.

2. **Three jet production**

The process $e^+e^- \to 3$ jets can be thought of as a two-step process, $e^+e^- \to \gamma^*$ followd by $\gamma^* \to q\bar{q}g$. Here γ^* is an off-shell photon.

(i) At leading order the diagrams for $\gamma^* \to q\bar{q}g$ give

$-i\mathcal{M}_{\gamma^* \to q\bar{q}g} =$



Compute $\langle |\mathcal{M}_{\gamma^* \to q\bar{q}g}|^2 \rangle$. You should average over the photon spin and sum over the spins, colors, and quark flavors in the final state. A few tips:

- You should allow the photon to be off-shell, $q^2 \neq 0$. However for simplicity you can take the other particles to be massless, $k_i^2 = 0$.
- You can sum over the photon and gluon spins using[1]

$$\sum_{\text{polarizations}} \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu} \,.$$

[1] You can't always perform gluon spin sums in such a simple way. See Peskin & Schroeder p. 159 and (17.73).

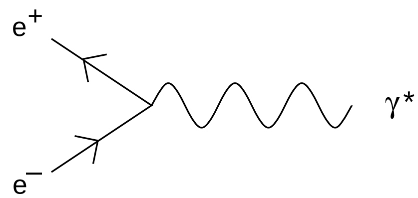
- To average over the photon spin you should divide your result by 3 for the three possible polarizations of a massive vector.
- You can sum over colors using $\mathrm{Tr}\lambda^a\lambda^b = 2\delta^{ab}$.
- You should express your answer in terms of the kinematic variables

  $$x_i = \frac{2k_i \cdot q}{q^2}\,.$$

  In the center of mass frame x_i is twice the energy fraction carried by particle i, $x_i = 2E_i/E_{\mathrm{cm}}$. Note that $x_1+x_2+x_3 = 2$.

(ii) Compute the spin-averaged $|\text{amplitude}|^2$ for $e^+e^- \to \gamma^*$ from

$$-i\mathcal{M}_{e^+e^-\to\gamma^*} \quad =$$



(iii) The spin-averaged $|\text{amplitude}|^2$ for the whole process is

$$\langle|\mathcal{M}|^2\rangle = \langle|\mathcal{M}_{e^+e^-\to\gamma^*}|^2\rangle \cdot \frac{1}{s^2} \cdot \langle|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2\rangle$$

where the $1/s^2$ in the middle is from the intermediate photon propagator.[2] Plug this into the cross-section formula

$$\frac{d\sigma}{dx_1 dx_2} = \frac{1}{256\pi^3}\langle|\mathcal{M}|^2\rangle$$

and find the differential cross-section for 3-jet events. You should reproduce Peskin & Schroeder (17.18).

[2]For a justification of this formula, including the factor of 3 for averaging over photon spins, see Peskin & Schroeder p. 261.